British Mathematical Olympiad

Round 1 : Wednesday, 12 January 2000

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

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1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

2. Show that, for every positive integer n,
$$121^n - 25^n + 1900^n - (-4)^n$$
is divisible by 2000.

3. Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that
$$AP + BP + CP$$
is minimized.

4. For each positive integer $k > 1$, define the sequence $\{a_n\}$ by
$$a_0 = 1 \quad \text{and} \quad a_n = kn + (-1)^n a_{n-1} \qquad \text{for each} \quad n \geq 1.$$
Determine all values of k for which 2000 is a term of the sequence.

5. The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy & Grumpy, and one of Bashful & Sneezy. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)
Suppose Snow-White agreed to take part as well. In how many ways could the four teams then be formed?